**University of California, Berkeley**
**Department of Mechanical Engineering**
**ME 170, Spring 2017**

**Homework 7**

## Problem 1

Recall that a particle moving under the influence of a central force has a constant areal velocity

$$\dot{A} = \frac{h}{2m}, \tag{1}$$

where $h/m$ is the angular momentum per unit mass. Also,

$$\left(\frac{h}{m}\right)^2 = GM\ell, \tag{2}$$

where $\ell$ is the semilatus rectum of the orbit. For an elliptical orbit, with semimajor axis $a$, semiminor axis $b$, and eccentricity $\varepsilon < 1$,

$$\ell = a(1-\varepsilon^2), \qquad b^2 = a^2(1-\varepsilon^2). \tag{3}$$

**(a)** The orbital period $\tau$ is given by

$$\tau = \frac{A(\tau)}{\dot{A}}. \tag{4}$$

Deduce that

$$\tau^2 = \frac{4\pi^2}{GM}a^3, \tag{5}$$

which is a statement of Kepler's third law.

**(b)** Examine how well (5) holds for the solar system. Complete the table below by obtaining the period $\tau$ and semimajor axis $a$ of each planet's orbit from the given data sheet. Then calculate $\tau^2/a^3$ and compare it with $4\pi^2/GM_S$ ($M_S = 1.9884 \times 10^{30}$ kg, $G = 6.674 \times 10^{-11}$ m$^3$/(kg · s$^2$)).

| Planet | Orbital Period $\tau$ (s) | Semimajor Axis $a$ (m) | $\tau^2/a^3$ | % error |
|---|---|---|---|---|
| Mercury | | | | |
| Venus | | | | |
| Earth | | | | |
| Mars | | | | |
| Jupiter | | | | |
| Saturn | | | | |
| Uranus | | | | |
| Neptune | | | | |

**(c)** Repeat Part (b) for Jupiter's four largest (Galilean) moons. The mass of Jupiter is $M_J = 1.8985 \times 10^{27}$ kg.

| Moon | Orbital Period $\tau$ (s) | Semimajor Axis $a$ (m) | $\tau^2/a^3$ | % error |
|---|---|---|---|---|
| Io | | | | |
| Europa | | | | |
| Ganymede | | | | |
| Callisto | | | | |

**Problem 2**

In 2008, astronomers discovered extrasolar planets orbiting the young star HR 8799, which is located 129 light years away from earth. The mass of the star is

$$M_H = 1.56\ M_S.$$

For the four planets in the system, the semi-major orbital axes are

$$\begin{aligned} \text{HR 8799 a}: &\quad 68.0 \text{ au} \\ \text{HR 8799 b}: &\quad 42.9 \text{ au} \\ \text{HR 8799 c}: &\quad 27.0 \text{ au} \\ \text{HR 8799 d}: &\quad 14.5 \text{ au}, \end{aligned}$$

where 1 au = $149.598 \times 10^9$ m (1 light year = 63241 au). Use Kepler's third law,

$$\frac{\tau^2}{a^3} = \frac{4\pi^2}{GM_H} \quad \text{s}^2/\text{m}^3 \tag{6}$$

to calculate the periods of these planets in years (1 year = 365.25 days = 31.5576 $\times 10^6$ s).

**Problem 3**

Consider a satellite orbiting the earth in a circular orbit $O_1$ of altitude 6000 km. Take the earth's mean radius $R$ to be 6378 km and the gravitational parameter $GM_E$ to be $398.6 \times 10^{12}$ $\text{m}^3/\text{s}^2$. Also, recall that for Kepler orbits, the specific angular momentum $h/m$ of the motion is related to the semilatus rectum $\ell$ of the orbit by

$$\left(\frac{h}{m}\right)^2 = GM_E \ell, \tag{7}$$

while the specific energy $E/m$ of the motion is related to the semilatus rectum $l$ and the orbital eccentricity $\varepsilon$ by

$$\frac{2\,(E/m)}{GM_E} = \frac{\varepsilon^2 - 1}{\ell}. \tag{8}$$

**(a)** Show that for a circular orbit of radius $r_0$, the satellite speed $v_0$ satisfies the relation

$$v_0^2 = \frac{GM_E}{r_0}, \tag{9}$$

and the specific energy is given by

$$\frac{E}{m} = -\frac{1}{2}\frac{GM_E}{r_0}. \tag{10}$$

**(b)** Calculate the quantities $h_1/m$ and $E_1/m$ for the circular orbit $O_1$.

**(c)** Suppose that at a point $A$ on the orbit $O_1$ the speed of the satellite is increased due to a tangential impulsive thrust by an amount

$$\Delta v_A = 660 \text{ m/s}. \tag{11}$$

Let $v'_A = v_A + \Delta v_A$. Calculate the dynamical quantities $h_2/m$ and $E_2/m$ for the new orbit $O_2$. Show that it is elliptical. Denote its apogee by $B$. Calculate the semilatus rectum $\ell_2$ and the eccentricity $\varepsilon_2$ of $O_2$. Also, calculate the semimajor and semiminor axes of the orbit, as well as the distances $r_{p2}$ and $r_{a2}$ to perigee and apogee, respectively. Using conservation of angular momentum, calculate the speed $v_B$ of the satellite at apogee. Sketch the orbits $O_1$ and $O_2$.

**(d)** Next, let the speed of the satellite be impulsively decreased at apogee by 200 m/s:

$$\Delta v_B = -200 \text{ m/s}. \tag{12}$$

Denote the new speed of the satellite by $v'_B$, and the new orbit by $O_3$. Determine the orbital parameters for $O_3$; use a subscript to identify them. Add the new orbit to your sketch. Denote its perigee by $C$. Calculate the satellite's speed $v_C$ at perigee.

**(e)** Argue that by reversing the increments (12) and (11), at $B$ and $A$, the satellite could be returned to its original circular orbit $O_1$ at a speed-increment cost of $\Delta v = 860$ m/s.

**(f)** As an alternative way to return to $O_1$, a Hohmann transfer semiellipse $O_4$ may be constructed with perigee at $C$ and apogee at a point $D$ that lies on the circle $O_1$ and is diametrically opposite to $A$. Thus,

$$r_{4p} = r_C = r_{3p}, \qquad r_{4a} = r_D = 12.378 \times 10^6 \text{ m}. \tag{13}$$

Calculate the quantities $a_4$, $\varepsilon_4$, $\ell_4$, and $b_4$ for the transfer orbit. Then, use Eqns. (7) and (8) to determine $h_4/m$ and $E_4/m$.

**(g)** Use the value $h_4/m$ to calculate the satellite's speed $v'_C$ in the orbit $O_4$, after the impulse at $C$. Likewise, calculate the speed $v_D$ which it has at apogee $D$, before the final impulse that returns it to the circular orbit $O_4$.

**(h)** Sum up the absolute values of the speed increments in Part (g) and compare the cost to that in Part (e).

## Problem 4

Suppose that an intercontinental ballistic missile is launched from the earth's surface with a speed $v_0 = 6.7$ km/s and a flight-path angle $\phi_0 = 20°$. The radius of the earth is 6378 km.

**(a)** Use the initial data to determine the dynamical constants $h/m$ and $E/m$ of the missile's orbit.

**(b)** Apply Eqns. (7) and (8) to calculate the semilatus rectum and eccentricity of the orbit.

**(c)** Calculate the semimajor and semiminor axes.

**(d)** Find the apogee and perigee.

**(e)** Calculate the speed of the missile at apogee.

**(f)** Recall that the orbit is described by the equation

$$r = \frac{\ell}{1 + \varepsilon \cos\theta}, \tag{14}$$

where the angle $\theta$ is the true anomaly. Calculate the value $\theta_0$ of $\theta$ at launch.

**(g)** Calculate the maximum altitude and range of the missile.

**(h)** Sketch the missile's orbit in relation to the earth.

## Problem 5

Consider an attractive central force field of the type

$$\mathbf{F} = -f(r)\mathbf{e}_r, \quad f > 0. \tag{15}$$

The angular momentum and energy integrals are given by

$$h = mr^2\dot{\theta} \quad (> 0) \tag{16}$$

and

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}\right) + V = E. \tag{17}$$

Let

$$u = \frac{1}{r} \tag{18}$$

and note that

$$\dot{r} = \frac{dr}{du}\frac{du}{d\theta}\,\dot{\theta} = -\frac{h}{m}\frac{du}{d\theta}, \quad \ddot{r} = -\left(\frac{h}{m}\right)^2 u^2\,\frac{d^2u}{d\theta^2}. \tag{19}$$

The equation of motion for $r$, namely

$$\ddot{r} - \left(\frac{h}{m}\right)^2 \frac{1}{r^3} + \frac{f}{m} = 0, \tag{20}$$

may then be expressed as

$$\frac{d^2u}{d\theta^2} + u - \frac{f/m}{(h/m)^2}\frac{1}{u^2} = 0. \tag{21}$$

Suppose that the law of attraction is that of inverse cube, i.e.,

$$\frac{f}{m} = \frac{\mu}{r^3}, \quad (\mu > 0). \tag{22}$$

**(a)** Calculate the corresponding potential energy function $V(r)$, taking $V \to 0$ as $r \to \infty$.

**(b)** Deduce that

$$\ddot{r} - \left[(\frac{h}{m})^2 - \mu\right]\frac{1}{r^3} = 0, \tag{23}$$

$$\frac{d^2u}{d\theta^2} + \left[1 - \frac{\mu}{(h/m)^2}\right]u = 0, \tag{24}$$

and

$$\left(\frac{du}{d\theta}\right)^2 + \left[1 - \frac{\mu}{(h/m)^2}\right]u^2 = \frac{2E/m}{(h/m)^2}. \tag{25}$$

**(c)** First, consider the case

$$\left(\frac{h}{m}\right)^2 - \mu = 0. \tag{26}$$

Deduce that

$$\ddot{r} = 0, \quad \frac{d^2u}{d\theta^2} = 0, \tag{27}$$

and show that the orbit must be of the form

$$\frac{1}{r} = u = A\theta + B, \tag{28}$$

where $A$ and $B$ are constants of integration. Further, argue that

$$\frac{E}{m} = \frac{1}{2}(\frac{h}{m})^2 A^2 \geq 0. \tag{29}$$

What is the special case $E = 0$? If $E > 0$, prove that the orbit cannot have any apsis. Sketch the particular orbit

$$r = \frac{1}{1 + \frac{\theta}{10}}. \tag{30}$$

**(d)** Next, consider the case

$$\left(\frac{h}{m}\right)^2 - \mu > 0. \tag{31}$$

Equation (24) is then if the form

$$\frac{d^2u}{d\theta^2} + \omega^2 u = 0. \tag{32}$$

Deduce that the orbits are described by

$$\frac{1}{r} = u = A\cos\omega\theta + B\sin\omega\theta = C\cos(\omega\theta + \psi), \tag{33}$$

where $A$, $B$, $C$, $\psi$ are constants. Relate $C$ to the dynamical constants and deduce that $E > 0$ for this case. Also, show that there is only one apsidal distance, which is given by

$$\frac{1}{r} = \frac{(h/m)^2 - \mu}{E/m}. \tag{34}$$

For the particular case

$$\frac{1}{r} = u = \cos 4\theta, \tag{35}$$

Find the values of $\theta$ at which the apsides occur and also solve for the apsidal distance. Sketch the orbits.

**(e)** Finally, consider the case

$$\frac{d^2u}{d\theta^2} - \mu < 0. \tag{36}$$

Equation (24) is then of the form

$$\frac{d^2u}{d\theta^2} - q^2\theta = 0. \tag{37}$$

Show that the orbits are of the form

$$\frac{1}{r} = u = Ae^{q\theta} + Be^{-q\theta}, \tag{38}$$

where $A$ and $B$ are constants.

(For the inverse cube law, the orbits are known collectively as Cotes's spirals.)